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NON-LINEAR EFFECTS IN A PLASMA

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8 January 1963

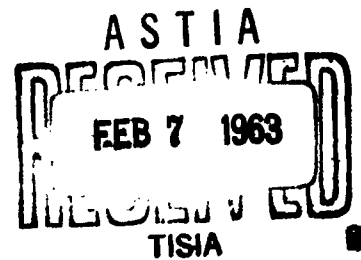
The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology, with the joint support of the U. S. Army, Navy and Air Force under Air Force Contract AF 19(628)-500.

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Abstract

In this report we study the effects of the non-linearities in the dispersion and attenuation of longitudinal waves in a plasma. We first work with the hydrodynamic equations and show that to a first approximation non-linearities affect the dispersion equation. We then work with the Boltzmann-Vlasov equation and indicate a general method for the study of the effects of non-linearities. Because of mathematical difficulties, we develop a hybrid formulation which is based on both the hydrodynamic and the Boltzmann-Vlasov equations and indicate that to a first approximation non-linearities affect both the dispersion equation and the attenuation in a plasma.



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I. Introduction

By linearizing the Boltzmann-Vlasov (B-V) or the hydrodynamic (H) equations that describe the longitudinal properties of a plasma, in effect we disregard interactions between modes of vibration in the system. If accounted for, these interactions could lead to attenuation of an excitation (density wave) in the plasma in addition to the attenuation due to collisions or to Landau damping; they could further lead to a modification of the plasma dispersion relation which, in the linearized case, is given by

$$\omega^2 = \omega_p^2 + (3kT/m) k^2 ; \text{ (B-V)} \quad (1)$$

$$\omega = \omega_p ; \text{ (H)} \quad (2)$$

The added attenuation can be attributed to the energy lost by a propagating density wave for the excitation of harmonics. On the other hand, the modification of the dispersion relation can be attributed to a correction of the longitudinal dielectric coefficient of the plasma because of non-linearities.

Some attempts have been made in the last few years to study the non-linear effects in a plasma in equilibrium described by the Boltzmann-Vlasov equation^(1, 2, 3). The little progress made consists perhaps in the realization of the mathematical and conceptual difficulties, but has added no new insight to the problem. In the present paper we make a renewed effort to understand the problem by attempting to describe the non-linear effects as a scattering process. In Sec. II we limit the discussion to the hydrodynamic model in order to keep a simple mathematical structure. We are aware that the hydrodynamic model is inadequate for large wave numbers k because it does not account for

the interaction of the collective mode with the free particles (leading to Landau damping). Although this shortcoming does not affect the validity of our method, it does cast some skepticism on the accuracy of the results. In principle, the method can be generalized to treat the Vlasov plasma. This generalization is developed in Sec. III.

II. Non-Linear Hydrodynamic Equations for a Plasma

We use the following set of equations to study the non-linearities in a homogeneous model of a plasma⁽⁴⁾:

(a) Continuity Equation:

$$\frac{\partial n(\bar{x}t)}{\partial t} + \nabla_{\bar{x}} \cdot [n(\bar{x}t) \bar{V}(\bar{x}t)] = 0 \quad , \quad (3)$$

(b) Equation of Motion:

$$\frac{\partial \bar{V}(\bar{x}t)}{\partial t} + [\bar{V}(\bar{x}t) \cdot \nabla_{\bar{x}}] \bar{V}(\bar{x}t) + \frac{e}{m} \nabla \phi(\bar{x}t) = 0 \quad , \quad (4)$$

(c) Poisson Equation:

$$\nabla^2 \phi(\bar{x}t) = -4\pi e \delta n(\bar{x}t) \quad . \quad (5)$$

The charge of the electron has been taken as $-e$ and the mean electron density n_0 is assumed to be neutralized by a positive sea of ions. $\delta n(\bar{x}t)$ is the excess density at a point \bar{x} and time t and therefore

$$n(\bar{x}t) = n_0 + \delta n(\bar{x}t) = n_0 \left\{ 1 + \frac{\delta n(\bar{x}t)}{n_0} \right\} = n_0 \{ 1 + \psi(\bar{x}t) \}$$

In terms of $\psi(\bar{x}t)$, Eqs. (3) and (5) become

$$\frac{\partial \psi(\bar{\mathbf{x}}t)}{\partial t} + \nabla_{\bar{\mathbf{x}}} \cdot \bar{\mathbf{V}}(\bar{\mathbf{x}}t) + \nabla_{\bar{\mathbf{x}}} \cdot [\psi(\bar{\mathbf{x}}t) \bar{\mathbf{V}}(\bar{\mathbf{x}}t)] = 0 \quad (3a)$$

$$\nabla^2 \phi(\bar{\mathbf{x}}t) = -4\pi en_o \psi(\bar{\mathbf{x}}t) \quad (5a)$$

By taking the Fourier transforms in time and space of Eqs. (3a), (4), and (5a), we obtain

$$-i\omega \psi(\bar{\mathbf{k}}\omega) + i\bar{\mathbf{k}} \cdot \bar{\mathbf{V}}(\bar{\mathbf{k}}\omega) + i\bar{\mathbf{k}} \cdot \int \bar{\mathbf{V}}(\bar{\mathbf{k}}'\omega') \psi(\bar{\mathbf{k}}-\bar{\mathbf{k}}', \omega-\omega') \frac{d\bar{\mathbf{k}}' d\omega'}{(2\pi)^4} = 0 \quad (6)$$

$$-i\omega \bar{\mathbf{V}}(\bar{\mathbf{k}}\omega) + i\bar{\mathbf{k}} \frac{e}{m} \phi(\bar{\mathbf{k}}\omega) + i \int (\bar{\mathbf{k}}-\bar{\mathbf{k}}') \cdot \bar{\mathbf{V}}(\bar{\mathbf{k}}'\omega') \bar{\mathbf{V}}(\bar{\mathbf{k}}-\bar{\mathbf{k}}', \omega-\omega') \frac{d\bar{\mathbf{k}}' d\omega'}{(2\pi)^4} = 0 \quad (7)$$

$$-k^2 \phi(\bar{\mathbf{k}}\omega) + 4\pi en_o \psi(\bar{\mathbf{k}}\omega) = 0 \quad , \quad (8)$$

where the quantities in the integrals represent the non-linear contributions.

The linearized version of the hydrodynamic equations have the form

$$-i\omega \psi(\bar{\mathbf{k}}\omega) + i\bar{\mathbf{k}} \cdot \bar{\mathbf{V}}(\bar{\mathbf{k}}\omega) = 0 \quad (9)$$

$$-i\omega \bar{\mathbf{V}}(\bar{\mathbf{k}}\omega) + i\bar{\mathbf{k}} \frac{e}{m} \phi(\bar{\mathbf{k}}\omega) = 0 \quad (10)$$

$$-k^2 \phi(\bar{\mathbf{k}}\omega) + 4\pi en_o \psi(\bar{\mathbf{k}}\omega) = 0 \quad (11)$$

and yield after some trivial substitutions

$$k^2 \left\{ 1 - \frac{\omega_p^2}{\omega^2} \right\} \phi(\bar{k}\omega) = 0 \quad . \quad (12)$$

Equation (12) can be interpreted either as the source-free Poisson equation in matter

$$k^2 \epsilon^L(\omega) \phi(\bar{k}\omega) = 0 \quad , \quad (13)$$

where ϵ^L is the longitudinal dielectric constant, or as the solution of the second order differential equation

$$\ddot{\phi}(\bar{x}t) + \omega_p^2 \phi(\bar{x}t) = 0 \quad ,$$

i. e. , the oscillator equation for the hydrodynamic oscillations in the plasma.

Returning now to the non-linear equations, we first solve Eq. (6) for $\psi(\bar{k}\omega)$ and substitute the value of $\bar{V}(\bar{k}\omega)$ from Eq. (7) to obtain

$$\begin{aligned} \psi(\bar{k}\omega) = & \frac{k_e^2}{m\omega^2} \phi(\bar{k}\omega) + \frac{1}{\omega^2} \int \bar{k} \cdot \bar{V}(\bar{k}-\bar{k}', \omega-\omega') (\bar{k}-\bar{k}') \cdot \bar{V}(\bar{k}'\omega') \frac{d\bar{k}'d\omega'}{(2\omega)^4} \\ & + \frac{1}{\omega} \int \bar{k} \cdot \bar{V}(\bar{k}'\omega') \psi(\bar{k}-\bar{k}', \omega-\omega') \frac{d\bar{k}'d\omega'}{(2\pi)^4} \end{aligned} \quad (14)$$

Further substitution of Eq. (14) in the Poisson equation finally gives

$$\begin{aligned}
k^2 \left\{ 1 - \frac{\omega_p^2}{\omega^2} \right\} \phi(\bar{k}\omega) &= \frac{4\pi en_o}{\omega^2} \int \bar{k} \cdot \bar{V}(\bar{k}-\bar{k}', \omega-\omega') (\bar{k}-\bar{k}') \cdot \bar{V}(\bar{k}'\omega') \frac{d\bar{k}' d\omega'}{(2\omega)^4} \\
&+ \frac{4\pi en_o}{\omega} \int \bar{k} \cdot \bar{V}(\bar{k}'\omega') \psi(\bar{k}-\bar{k}', \omega-\omega') \frac{d\bar{k}' d\omega'}{(2\omega)^4} \quad (15)
\end{aligned}$$

To bring Eq. (15) in a more lucid form, we substitute the \bar{V} 's inside the integrals with their values given by Eqs. (6) and (7). Thus

$$\begin{aligned}
k^2 \left\{ 1 - \frac{\omega_p^2}{\omega^2} \right\} \phi(\bar{k}\omega) &= \left(\frac{e}{m} \right)^2 \frac{4\pi en_o}{\omega^2} \int \frac{\bar{k} \cdot (\bar{k}-\bar{k}')}{\omega-\omega'} \frac{\bar{k} \cdot (\bar{k}-\bar{k}')}{\omega'} \phi(\bar{k}-\bar{k}', \omega-\omega') \phi(\bar{k}'\omega') \frac{d\bar{k}' d\omega'}{(2\pi)} \\
&+ \frac{4\pi e^2 n_o}{m\omega} \int \frac{\bar{k} \cdot \bar{k}'}{\omega'} \phi(\bar{k}'\omega') \psi(\bar{k}-\bar{k}', \omega-\omega') \frac{d\bar{k}' d\omega'}{(2\pi)^3} \quad (16) \\
&+ \dots
\end{aligned}$$

The terms that have been neglected could be further reduced by successive substitutions to give higher order terms; however, the algebra is prohibitive. In any case we can clearly see that Eq. (16) is an integral equation for the potential $\phi(k\omega)$ and its right-hand side could be interpreted as a source term for the Poisson equation in matter, i. e.,

$$k^2 \left\{ 1 - \frac{\omega_p^2}{\omega^2} \right\} \phi(k\omega) = 4\pi\rho(k\omega) \quad . \quad (17)$$

Our problem now is to find a solution for the integral equation. Since $\rho(\bar{k}\omega)$ is expected to be a small quantity, it becomes obvious that we should seek solutions close to the ones given by the linear hydrodynamic model. According to the model, density oscillations in the plasma can occur at only one frequency ω and at an arbitrary wave number \bar{k} . We know, however, from the solution of the B-V equation that this is only approximately true; i. e., the oscillation frequency is given by $\omega = \omega_p(\bar{k})$ except in the limit of $k \rightarrow 0$ where $\omega \doteq \omega_p$. Selecting therefore an arbitrary but small wave number \bar{k}_p , we assume that at a first approximation the potential in the hydrodynamic case is given by

$$\phi(k\omega) = \phi_0 (2\pi)^4 \delta(k - k_p) \delta(\omega - \omega_p) \quad , \quad (18)$$

where ϕ_0 is the amplitude of the potential. However, the integral equation 16 expresses the fact that if non-linear terms are accounted for there are contributions to the potential from all \bar{k}' and ω' . Stated another way, non-linearities produce a slipping of the frequency and wave number, a process very similar to incoherent scattering. In fact, if a plasma oscillation is to be viewed as a density wave propagating in the plasma, then it becomes clear that it can be scattered out of the direction it is propagating. As the wave is scattered out of the forward direction it loses a part of its energy. As a first approximation we assume the attenuation to be small and separate $\phi(\bar{k}\omega)$ in two parts: the coherent part consisting of the propagating density wave at $\omega_p - \bar{k}_p$, and the incoherent part $\psi^{(1)}(\bar{k}\omega)$:

$$\phi(\bar{k}\omega) = \phi_0 (2\pi)^4 \delta(\bar{k} - \bar{k}_p) \delta(\omega - \omega_p) + \psi^{(1)}(\bar{k}\omega) \quad . \quad (19)$$

We introduce now Eq. (19) in Eq. (16) and obtain

$$\begin{aligned}
& k^2 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \phi_o (2\pi)^4 \delta(\omega - \omega_p) \delta(\bar{k} - \bar{k}_p) + k^2 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \phi^{(1)}(\bar{k}\omega) = \\
& \frac{\omega_p^2}{\omega^2} \phi_o \int \frac{\omega}{\omega'} \bar{k} \cdot \bar{k}' (2\pi)^4 \delta(\omega' - \omega_p) \delta(\bar{k}' - \bar{k}_p) \psi(\bar{k} - \bar{k}', \omega - \omega') \frac{d\bar{k}' d\omega'}{(2\pi)^4} \\
& + \frac{\omega_p^2}{\omega^2} \int \frac{\omega \bar{k} \cdot \bar{k}'}{\omega'} \phi^{(1)}(\bar{k}'\omega') \psi(\bar{k} - \bar{k}', \omega - \omega') \frac{d\bar{k}' d\omega'}{(2\pi)^4} + \mathcal{O}, \tag{20}
\end{aligned}$$

where \mathcal{O} consists of terms contributed by the first integral in Eq. (16). We can see by inspection that these terms are both small (since they are proportional to ϕ_o^2 , $\phi^{(1)}\phi_o$, and $\phi^{(1)}\phi^{(1)}$) and incoherent with respect to ϕ_o and $\phi^{(1)}(\bar{k}\omega)$. We will therefore neglect them. From Eq. (20) we clearly have

$$\begin{aligned}
k^2 \left(1 - \frac{\omega_p^2}{\omega^2}\right) \phi^{(1)}(\bar{k}\omega) &= \phi_o \frac{\omega_p^2}{\omega^2} - \frac{\omega}{\omega_p} \bar{k} \cdot \bar{k}_p \psi(\bar{k} - \bar{k}_p, \omega - \omega_p) \\
&= \phi_o \frac{\omega_p}{\omega} \bar{k} \cdot \bar{k}_p \frac{\mathcal{S}n(\bar{k} - \bar{k}_p, \omega - \omega_p)}{n_o} \tag{21}
\end{aligned}$$

and therefore

$$\phi^{(1)}(\bar{k}\omega) = \frac{\phi_o \omega \omega_p (\bar{k} \cdot \bar{k}_p)}{k^2 (\omega^2 - \omega_p^2)} \cdot \frac{\mathcal{S}n(\bar{k} - \bar{k}_p, \omega - \omega_p)}{n_o} \tag{22}$$

where the division by $\omega^2 - \omega_p^2$ will be interpreted as follows. We first expand

$$\frac{1}{\omega^2 - \omega_p^2} = \frac{1}{2\omega_p} \left(\frac{1}{\omega - \omega_p} - \frac{1}{\omega + \omega_p} \right)$$

and then write

$$\lim_{\alpha \rightarrow 0} \frac{1}{\omega - \omega_p \pm i\alpha} = \mathcal{P} \frac{1}{\omega - \omega_p} \mp \pi i \delta(\omega - \omega_p) \quad (23)$$

along with a similar expression for $(\omega + \omega_p)^{-1}$. This corresponds to the assumption of a negligibly small loss term in either the dielectric constant $\epsilon(\omega)$ or the equation of motion for $\phi(\vec{x}t)$. Inserting Eq. (22) back in Eq. (20), we obtain

$$k^2 \epsilon(\omega) \phi_0 (2\pi)^4 \delta(\omega - \omega_p) \delta(\vec{k} - \vec{k}_p) = \frac{\omega_p^2}{\omega} \int \phi_0 \frac{(\vec{k} \cdot \vec{k}')(\vec{k} \cdot \vec{k}_p)}{k'^2} \frac{\omega_p}{\omega'^2 - \omega_p^2} \frac{\delta n(\vec{k}' - \vec{k}_p, \omega' - \omega_p) \delta n(\vec{k} - \vec{k}', \omega - \omega')}{n_0^2} \frac{d\vec{k}' d\omega'}{(2\pi)^4} \quad (24)$$

This equation can be interpreted as an equation describing the scattering of a coherent density wave at ω_p , \vec{k}_p from density waves traveling at directions \vec{k}' and frequencies ω' . It is clear that the density deviations $\delta n(\vec{k}' - \vec{k}_p, \omega' - \omega_p)$, $\delta n(\vec{k} - \vec{k}', \omega - \omega')$ inside the integral of Eq. (24) represent density waves with respective wave numbers and frequencies $\vec{k}' - \vec{k}_p, \omega' - \omega_p$; $\vec{k} - \vec{k}', \omega - \omega'$. However, it is also clear from the left-hand side of Eq. (24) that we are interested in the case where $\vec{k} = \vec{k}_p$, $\omega = \omega_p$, i. e., in the forward direction. (See Fig. 1.)

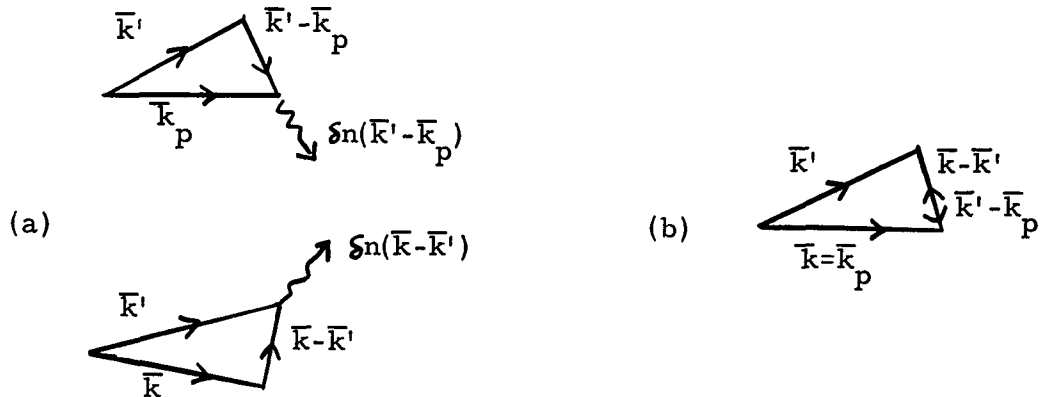


Fig. 1 Density Waves

This corresponds to imposing the conditions*

$$(\bar{k}' - \bar{k}_p) + (\bar{k} - \bar{k}') = 0 \quad \text{i. e. , } \bar{k} = \bar{k}_p \quad (25)$$

$$(\omega' - \omega_p) + (\omega - \omega') = 0 \quad \text{i. e. , } \omega = \omega_p \quad (26)$$

Taking now the average on both sides of Eq. (24) and noting⁽⁵⁾ that

$$\overline{\delta n(\bar{k}' - \bar{k}_p, \omega' - \omega_p) \delta n(\bar{k} - \bar{k}', \omega - \omega')} = (2\pi)^4 \langle |\delta n(\bar{k}' - \bar{k}_p, \omega' - \omega_p)|^2 \rangle \delta(\bar{k} - \bar{k}_p) \delta(\omega - \omega_p)$$

and that $\langle \phi_o \rangle = \phi_o$, we derive from Eq. (24)

$$k^2 [\epsilon(\omega) + \delta \epsilon(k\omega)] (2\pi)^4 \phi_o \delta(\bar{k} - \bar{k}_p) \delta(\omega - \omega_p) = 0$$

where

$$\delta \epsilon(\bar{k}\omega) = - \frac{1}{k^2} \frac{\omega_p}{\omega} \int \frac{(\bar{k} \cdot \bar{k}')^2}{k'^2} \frac{\omega_p^2}{\omega'^2 - \omega_p^2} \frac{\langle |\delta n(\bar{k}' - \bar{k}_p, \omega' - \omega_p)|^2 \rangle}{n_o^2} \frac{d\bar{k}' d\omega}{(2\pi)^4} \quad (27)$$

where now $\langle |\delta n|^2 \rangle$ is the mean square fluctuation of the density. We note that we cannot derive information concerning the amplitude of plasma oscillations from the hydrodynamic or the B-V equations. Nevertheless, it is the thermal motion in the microscopic scale that initiates the plasma oscillations in the macroscopic scale and therefore the amplitude of the oscillations can only be derived from thermodynamic reasoning. Appendix A is devoted to the evaluation of the fluctuation spectrum of a hydrodynamic plasma by making use of Nyquist's theorem. We find that

* If instead of density waves we were to consider particles, Eqs. (25) and (26) would represent momentum and energy conservation.

$$\langle |\delta n(\bar{k}\omega)|^2 \rangle = n_o k^2 \lambda_D^2 \frac{\omega_p}{2\omega} \{ \delta(\omega - \omega_p) - \delta(\omega + \omega_p) \} \quad , \quad (28)$$

where λ_D is the Debye length. Since the fluctuation spectrum has just two lines at $\pm \omega_p$ and no width, we are entitled to call the fluctuation a plasma oscillation. The fluctuation spectrum derived from the B-V equations has a finite width which, in effect, corresponds to Landau damping ; however, we can readily show that in the limit of $k\lambda_D \rightarrow 0$, the B-V spectrum is again given by Eq. (28). With the available information we can now compute the correction to the dielectric coefficient $\epsilon(\bar{k}\omega)$. Calling

$$\delta\epsilon = \delta\epsilon' + i\delta\epsilon''$$

$$\bar{K} = \bar{k}' - \bar{k}_p \quad , \quad \Omega = \omega' - \omega_p \quad ,$$

and using Eq. (23), we obtain

$$\delta\epsilon' = - \frac{1}{k^2} \frac{\omega_p^2}{\omega} \int \frac{(\bar{k}' \cdot \bar{k}_p)^2}{k'^2} \frac{1}{2} \mathcal{P} \left(\frac{1}{\omega' - \omega_p} - \frac{1}{\omega' + \omega_p} \right) \frac{\langle |\delta n(\bar{K}, \Omega)|^2 \rangle}{n_o^2} \frac{d\bar{k}' d\omega'}{(2\pi)^4} \quad (29)$$

$$\delta\epsilon'' = + \frac{1}{k^2} \frac{\omega_p^2}{\omega} \int \frac{(\bar{k}' \cdot \bar{k}_p)^2}{k'^2} \frac{\pi}{2} [\delta(\omega' - \omega_p) - \delta(\omega' + \omega_p)] \frac{\langle |\delta n(\bar{K}, \Omega)|^2 \rangle}{n_o^2} \frac{d\bar{k}' d\omega'}{(2\pi)^4} \quad (30)$$

where

$$\langle |\delta n(\bar{K}, \Omega)|^2 \rangle = n_o (\bar{k}' - \bar{k}_p)^2 \lambda_D^2 \frac{\omega_p}{(\omega' - \omega_p)} \frac{1}{2} \{ \delta(\omega' - 2\omega_p) - \delta(\omega') \} \quad (31)$$

Explicitly we have for $\delta\epsilon'$

$$\begin{aligned} \delta\epsilon' &= \int \frac{(\bar{k}' \cdot \bar{k}_p)^2 (\bar{k}' - \bar{k}_p)^2}{\bar{k}^2 \bar{k}'^2} \frac{d\bar{k}'}{(2\pi)^3} \left(\frac{\lambda_D^2}{2\pi n_o} - \frac{2}{3} \frac{\omega_p}{\omega} \right) \\ &= g(\bar{k}) \frac{\lambda_D^2 \omega_p}{\pi n_o^3 \omega} \end{aligned} \quad (32)$$

where $g(\bar{k})$ is the value of the integral between 0 and some k_{\max} .

The dispersion equation then becomes

$$\epsilon + \delta\epsilon' = 0 = 1 - \frac{\omega_p^2}{\omega^2} + g(\bar{k}) \left(\frac{\lambda_D^2}{3\pi n_0} \right) \frac{\omega_p}{\omega}$$

which gives

$$\begin{aligned} \omega^2 &= \omega_p^2 - g(\bar{k}) \frac{\lambda_D^2}{3\pi n_0} \left(\frac{\omega_p}{\omega} \right) \omega^2 \\ &\sim \omega_p^2 - g(\bar{k}) \frac{\lambda_D^2}{3\pi n_0} \omega_p^2 \\ &= \omega_p^2 - g(\bar{k}) \left(\frac{\kappa T}{m} \frac{1}{3n_0} \right) \end{aligned} \quad (33)$$

This implies that the frequency decreases because of non-linearities. As a result the group velocity

$$v_g = \frac{\partial \omega}{\partial k}$$

becomes negative. If, however, the vector \bar{k}' in Fig. 1 were to be reversed, the sign of $g(\bar{k})$ in Eq. (33) would also reverse and in this case

$$\omega^2 = \omega_p^2 + g(\bar{k}) \left(\frac{\kappa T}{m} \frac{1}{3n_0} \right)$$

As a result $v_g > 0$.

From Eq. (30) we find that $\delta\epsilon'' = 0$. As a consequence and to the approximation we are using, there is no attenuation in the propagating density wave. In conclusion we may say that, to first order, non-linearities affect only the dispersion equation.

III. Non-Linear Vlasov Equation

As mentioned in the previous section, the hydrodynamic model of the plasma is inadequate in the sense that it does not give a dispersion relation and a damping term in the linear approximation. It becomes therefore necessary to study the non-linear Vlasov equation in order to obtain a correction to the dispersion relation and to the damping. We outline here the method that could be useful in a program of this nature. However, we do not attempt to give an answer to the problem because the equations turn out to be very involved. Instead we use some of the results to improve the corresponding hydrodynamic formulation. Vlasov's equation for the distribution function $f(\vec{x}\vec{v}t)$ has the form

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla_{\vec{x}} f + \frac{e}{m} \nabla \phi \cdot \nabla_{\vec{v}} f = 0 \quad (34)$$

while the Poisson equation for a neutralized electron plasma is

$$\nabla^2 \phi = 4\pi e \int f_1(\vec{x}\vec{v}t) d\vec{v} \quad (f = f_0 + f_1) \quad (35)$$

Fourier transformation in space and time of both equations gives

$$(-i\omega + i\vec{k} \cdot \vec{v}) f_1(\vec{k}\omega\vec{v}) + i \frac{e}{m} \vec{k} \cdot \nabla_{\vec{v}} f_0 \phi(k\omega) \\ + i \frac{e}{m} \int \phi(\vec{k}'\omega') \vec{k}' \cdot \nabla_{\vec{v}} f_1(\vec{k}-\vec{k}', \omega-\omega', \vec{v}) \frac{d\vec{k}' d\omega'}{(2\pi)^4} = 0$$

$$k^2 \phi(\vec{k}\omega) = -4\pi e \int f_1(\vec{k}\omega\vec{v}) d\vec{v} .$$

Solving for $f_1(\vec{k}\omega\vec{v})$, we obtain

$$f_1(\vec{k}\omega\vec{v}) = \frac{e}{m} \frac{\vec{k} \cdot \nabla_{\vec{v}} f_0}{\omega - \vec{k} \cdot \vec{v}} \phi(k\omega) + \frac{e}{m} \int \phi(\vec{k}'\omega') \frac{\vec{k}' \cdot \nabla_{\vec{v}} f_1(\vec{k}-\vec{k}', \omega-\omega', \vec{v})}{\omega - \vec{k} \cdot \vec{v}} \frac{d\vec{k}' d\omega'}{(2\pi)^4} \quad (36)$$

We integrate Eq. (36) over \bar{v} and insert in the Poisson equation

$$k^2 \left[1 + \frac{4\pi e^2}{mk} \int \frac{\bar{k} \cdot \nabla_{\bar{v}} f_0}{\omega - \bar{k} \cdot \bar{v}} d\bar{v} \right] \phi(\bar{k}\omega) =$$

$$\frac{4\pi e^2}{m} \int \phi(\bar{k}'\omega') \frac{\bar{k}' \cdot \nabla_{\bar{v}} f_1(\bar{k} - \bar{k}', \omega - \omega', \bar{v})}{\omega - \bar{k} \cdot \bar{v}} \frac{d\bar{k}' d\omega'}{(2\pi)^4} d\bar{v}.$$

The quantity in the square brackets is by definition the longitudinal dielectric coefficient $\epsilon^L(\bar{k}\omega)$, and therefore, we can now write the equivalent of Eq. (16)

$$k^2 \epsilon^L(\bar{k}\omega) \phi(\bar{k}\omega) = \frac{4\pi e^2}{m} \int \phi(\bar{k}'\omega') \frac{\bar{k}' \cdot \nabla_{\bar{v}} f_1(\bar{k} - \bar{k}', \omega - \omega', \bar{v})}{\omega - \bar{k} \cdot \bar{v}} \frac{d\bar{k}' d\omega'}{(2\pi)^4} d\bar{v} \quad (37)$$

$$= 4\pi \rho(\bar{k}\omega)$$

Equation (37) is an integral equation for $\phi(\bar{k}\omega)$, and it represents the same physical problem as the equivalent hydrodynamic equation. It presents, however, a mathematical problem of larger complexity because of the denominator $(\omega - \bar{k} \cdot \bar{v})$. To bring Eq. (37) in better correspondence to Eq. (16), we write

$$\int \frac{\bar{k}' \cdot \nabla_{\bar{v}} f_1(\bar{k}, \bar{\omega}, \bar{v})}{\omega - \bar{k} \cdot \bar{v}} d\bar{v} = \int \frac{\bar{k} \cdot \bar{k}' f_1(\bar{k}, \bar{\omega}, \bar{v})}{(\omega - \bar{k} \cdot \bar{v})^2} d\bar{v}.$$

It then becomes obvious that for $k \rightarrow 0$ we obtain Eq. (16). To proceed from Eq. (37) we could use an iteration procedure, or we could write as in Section 2

$$\phi(\bar{k}\omega) = \phi_0(2\pi)^4 \delta(\bar{k} - \bar{k}_p) \delta(\omega - \omega_p) + \phi^{(1)}(\bar{k}\omega) \quad (38)$$

$$\text{with } \omega^2 = \omega_p^2 + (3 \kappa T/m) k_p^2 = \omega_p^2 + v_o^2 k^2.$$

Then

$$\begin{aligned}
& k^2 \epsilon^L(k\omega) \phi_o (2\pi)^4 \delta(\bar{k}-\bar{k}_p) \delta(\omega-\omega_p) + k^2 \epsilon^L(\bar{k}\omega) \phi^{(1)}(\bar{k}\omega) \\
&= \frac{4\pi e^2}{m} \phi_o \int \frac{\bar{k}'_p \cdot \nabla_{\bar{v}} f_1(\bar{k}-\bar{k}'_p, \omega-\omega_p, \bar{v})}{\omega-\bar{k} \cdot \bar{v}} d\bar{v} \\
&+ \frac{4\pi e^2}{m} \int \phi^{(1)}(\bar{k}'\omega') \frac{\bar{k}' \cdot \nabla_{\bar{v}} f_1(\bar{k}-\bar{k}', \omega-\omega', \bar{v})}{\omega-\bar{k} \cdot \bar{v}} \frac{d\bar{k}' d\omega'}{(2\pi)^4} d\bar{v} ,
\end{aligned}$$

or finally (see Sec. II)

$$\begin{aligned}
& k^2 \epsilon^L(k\omega) \phi_o (2\pi)^4 \delta(\bar{k}-\bar{k}_p) \delta(\omega-\omega_p) = \\
& \left(\frac{4\pi e^2}{m}\right)^2 \phi_o \int \frac{[\bar{k}_p \cdot \nabla_{\bar{v}} f_1(\bar{k}'-\bar{k}_p, \omega'-\omega_p, \bar{v}')] [\bar{k}' \cdot \nabla_{\bar{v}} f_1(\bar{k}-\bar{k}', \omega-\omega', \bar{v})]}{k'^2 \epsilon^L(\bar{k}'\omega') (\omega'-\bar{k}' \cdot \bar{v}') (\omega-\bar{k} \cdot \bar{v})} d\bar{v} d\bar{v}' \frac{d\bar{k}' d\omega'}{(2\pi)^4} . \quad (39)
\end{aligned}$$

Our intention was to derive an equation of the form

$$k^2 [\epsilon^L(\bar{k}\omega) + \delta\epsilon(\bar{k}\omega)] \phi_o (2\pi)^4 \delta(\bar{k}-\bar{k}_p) \delta(\omega-\omega_p) ,$$

and from this the corrected dispersion relation $\epsilon'^L(k\omega) + \delta\epsilon'(k\omega) = 0$. However, we have not been able to make progress along these lines. As a hybrid solution to the problem we could use the left-hand side of Eq. (39) and the hydrodynamic expression for the right-hand side. From Eq. (27) we would then have

$$\delta\epsilon(\bar{k}\omega) \doteq - \frac{1}{k^2} \frac{\omega_p}{\omega} \int \frac{(\bar{k}_p \cdot \bar{k}')^2}{k'^2} \frac{1}{\epsilon^L(\bar{k}'\omega')} \frac{\langle |n(\bar{k}'-\bar{k}_p, \omega'-\omega_p)|^2 \rangle}{n_o^2} \frac{d\bar{k}' d\omega'}{(2\pi)^4} , \quad (40)$$

or since from the fluctuation theorem we have

$$\begin{aligned} \frac{\langle |\delta n(k\omega)|^2 \rangle}{n_o} &= \frac{kT}{\pi} \frac{k^2}{\omega e^2 4\pi n_o} \Im \left(-\frac{1}{\epsilon^L(\bar{k}\omega)} \right) \\ &= \frac{k^2 \lambda_D^2}{\pi \omega} \frac{\epsilon''^L(\bar{k}\omega)}{|\epsilon^L(\bar{k}\omega)|^2} \end{aligned}$$

Equation (40) becomes

$$\delta \epsilon(k\omega) \doteq -\frac{1}{k^2} \frac{\omega_p}{\omega} \int \frac{(\bar{k}_p \cdot \bar{k}')^2}{k'^2} \frac{\epsilon^L(\bar{k}'\omega') - i\epsilon''^L(\bar{k}'\omega')}{|\epsilon^L(\bar{k}', \omega')|^2} \frac{K^2 \lambda_D^2}{\pi \Omega} \frac{\epsilon''^L(\bar{K}\Omega)}{|\epsilon^L(\bar{K}\Omega)|^2} \frac{d\bar{k}' d\omega'}{(2\pi)^4} \quad (41)$$

If we use the hydrodynamic formula for the fluctuations, the integration over ω' can be performed with ease. The results are interesting for the imaginary part of $\delta \epsilon(k\omega)$:

$$\begin{aligned} \delta \epsilon''(k\omega) &= + \frac{1}{k^2} \frac{\omega_p}{\omega} \int \frac{(\bar{k}_p \cdot \bar{k}')^2}{k'^2} \frac{\epsilon''^L(\bar{k}'\omega')}{|\epsilon^L(\bar{k}'\omega')|^2} \frac{1}{n_o} (\bar{k}' - \bar{k}_p)^2 \lambda_D^2 \frac{\omega_p}{\omega' - \omega_p} \{ \delta(\omega' - 2\omega_p) - \delta(\omega') \} \frac{d\bar{k}' d\omega'}{(2\pi)^4} \\ &= \int_0^{k_D} \frac{(\bar{k}_p \cdot \bar{k}')^2 (\bar{k}' - \bar{k}_p)^2}{\bar{k}^2 \bar{k}'^2} \frac{d\bar{k}'}{(2\pi)^3} \left\{ \frac{\epsilon''^L(\bar{k}', 2\omega_p)}{|\epsilon^L(\bar{k}', 2\omega_p)|^2} + \frac{\epsilon''^L(\bar{k}', 0)}{|\epsilon^L(\bar{k}', 0)|^2} \right\} \frac{\omega_p}{\omega} \frac{\lambda_D^2}{n_o} \frac{1}{2\pi} \quad (42a) \end{aligned}$$

$$\begin{aligned} &= \sum_{\substack{k' < k_D \\ \omega = \omega_p}} \frac{(\bar{k}_p \cdot \bar{k}')^2 (\bar{k}' - \bar{k}_p)^2}{\bar{k}^2 \bar{k}'^2} \left[\frac{\epsilon''^L(\bar{k}', 2\omega_p)}{|\epsilon^L(\bar{k}', 2\omega_p)|^2} + \frac{\epsilon''^L(\bar{k}', 0)}{|\epsilon^L(\bar{k}', 0)|^2} \right] \frac{\lambda_D^2}{n_o} \frac{1}{2\pi} \quad (42b) \end{aligned}$$

where the integration or summation extends from $k=0$ to k_D , the Debye wave number. We note now that the fluctuations of the electric field in the plasma

are given by

$$\langle |\vec{E}(\vec{k}\omega)|^2 \rangle = \frac{8\kappa T}{\omega} \frac{\epsilon''^L(\vec{k}\omega)}{|\epsilon^L(\vec{k}\omega)|^2}, \quad (43)$$

and therefore, the energy is

$$\mathcal{E}(\vec{k}\omega) = \frac{\langle |\vec{E}(\vec{k}\omega)|^2 \rangle}{8\pi} = \frac{\kappa T}{\pi\omega} \frac{\epsilon''^L(\vec{k}\omega)}{|\epsilon^L(\vec{k}\omega)|^2}$$

Thus

$$\delta\epsilon''(\vec{k}\omega) = \sum_{\vec{k}' < k_D} \frac{(\vec{k}_p \cdot \vec{k}')^2 (\vec{k}' - \vec{k}_p)^2}{\vec{k}^2 \vec{k}'^2} \{ \mathcal{E}(\vec{k}', 2\omega_p) + \mathcal{E}(\vec{k}', 0) \} \frac{1}{\omega_p^m} \frac{1}{n_o} \quad (44)$$

where the summation is for wave numbers smaller than the Debye wave number.

The imaginary part of $\delta\epsilon$ gives a measure of the attenuation of the coherent density wave because of non-linearities in the system. Equation (44) can now be compared with a similar equation quoted by Pines.⁶ In the Pines picture this attenuation is expressed in terms of the rate of change of plasmons due to non-linear interactions. In addition to $\delta\epsilon''$ we could also compute $\delta\epsilon'$ to derive a correction for the B-V dispersion equation.

In conclusion, we may say that in the hydrodynamic model of a plasma only the dispersion equation is affected by non-linearities. The indication, however, is that in the B-V model of a plasma non-linearities introduce not only a correction to the dispersion equation but also an attenuation.

Appendix A: Calculation of the Density Fluctuation

If an external potential ϕ^{ext} is included in the linearized version of the hydrodynamic equations, we have from Eqs. (9-11)

$$-i\omega\psi(\vec{k}\omega) + i\vec{k} \cdot \vec{\nabla}(\vec{k}\omega) = 0 \quad (\text{A. 1})$$

$$-i\omega\vec{\nabla}(\vec{k}\omega) + i\vec{k} \frac{e}{m} \phi(\vec{k}\omega) + i\vec{k} \frac{e}{m} \phi^{\text{ext}}(\vec{k}\omega) = 0 \quad (\text{A. 2})$$

$$-k^2\phi(\vec{k}\omega) + 4\pi en_0\psi(\vec{k}\omega) = 0 \quad (\text{A. 3})$$

Multiplying (A. 2) by \vec{k} , introducing the result in (A. 1) and making use of the Poisson Equation we have successively

$$\begin{aligned} \psi(\vec{k}\omega) &= \omega \frac{\vec{k} \cdot \vec{\nabla}(\vec{k}\omega)}{\omega^2} \\ &= \frac{e}{m} \frac{k^2}{\omega^2} \phi(\vec{k}\omega) + \frac{e}{m} \frac{k^2}{\omega^2} \phi^{\text{ext}}(\vec{k}\omega) \\ &= \frac{4\pi e^2 n_0}{m\omega^2} \psi(\vec{k}\omega) + \frac{e}{m} \frac{k^2}{\omega^2} \phi^{\text{ext}}(\vec{k}\omega) \end{aligned}$$

Thus

$$(\omega^2 - \omega_p^2) \psi(\vec{k}\omega) = \frac{e}{m} k^2 \phi^{\text{ext}}(\vec{k}\omega)$$

or

$$\psi(\vec{k}\omega) = \frac{(e/m) k^2 \phi^{\text{ext}}(\vec{k}\omega)}{\omega^2 - \omega_p^2}$$

$\psi(\vec{k}\omega)$ is the linear response of the hydrodynamic plasma to an external potential ϕ^{ext} . Since $\psi(\vec{k}\omega) = \delta n(\vec{k}\omega)/n_0$, we have further

$$\rho(\vec{k}\omega) = e\delta n(\vec{k}\omega) = \frac{k^2}{4\pi} \frac{\omega_p^2}{\omega^2 - \omega_p^2} \phi^{\text{ext}}(\vec{k}\omega) = \frac{k^2}{4\pi} \frac{1 - \epsilon}{\epsilon} \phi^{\text{ext}}(\vec{k}\omega)$$

where $\epsilon = 1 - \omega_p^2 / \omega^2$. The current is given by

$$\bar{j}(\bar{k}\omega) = \frac{i\omega}{4\pi} \frac{\omega_p^2}{\omega^2 - \omega_p^2} \bar{E}^{\text{ext}}(\bar{k}\omega) \quad (\text{A. 5})$$

where $\bar{E}^{\text{ext}}(\bar{k}\omega) = -ik\phi^{\text{ext}}(\bar{k}\omega)$. From Nyquist's theorem we know that the mean square fluctuation for \bar{j} is

$$\langle |\bar{j}(\bar{k}\omega)|^2 \rangle = \frac{\kappa T}{\pi} \text{Re} \left\{ \frac{i\omega}{4\pi} \frac{\omega_p^2}{\omega^2 - \omega_p^2} \right\} \quad (\text{A. 6})$$

and therefore

$$\langle |\delta n(\bar{k}\omega)|^2 \rangle = \frac{k^2}{\omega_e^2} \langle |\bar{j}(\bar{k}\omega)|^2 \rangle$$

If we interpret the singularity of (A. 6) as

$$\frac{\omega_p^2}{\omega^2 - \omega_p^2} = \frac{\omega_p}{2} \mathcal{P} \left\{ \frac{1}{\omega - \omega_p} - \frac{1}{\omega + \omega_p} \right\} + \frac{\omega_p}{2} \frac{\pi}{i} \{ \delta(\omega - \omega_p) - \delta(\omega + \omega_p) \},$$

we readily obtain

$$\frac{\langle |\delta n(\bar{k}\omega)|^2 \rangle}{n_o} = \frac{\kappa T}{4\pi e^2 n_o} \frac{k^2 \omega_p}{2\omega} \{ \delta(\omega - \omega_p) - \delta(\omega + \omega_p) \} \quad (\text{A. 7})$$

which integrates to

$$\frac{1}{n_o} \int \langle |\delta n(\bar{k}\omega)|^2 \rangle d\omega = k^2 \lambda_D^2$$

where

$$\lambda_D^2 = \kappa T / 4\pi e^2 n_o$$

is the Debye length.

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